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Cyril Furtlehner*, Marc Schoenauer†

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Abstract: Message passing algorithms have been very successful in solving hard combinatorial problems, and have resulted in breakthrough results in the domain of random K-SAT problems. However, only single-objective SAT problems have been adressed by survey-propagation methods, whereas most real-world problems are indeed multi objective. A first approach to multi objective optimization using a message-passing algorithm is introduced, that aims at sampling the Pareto set, i.e. the set of Pareto-non-dominated solutions. Several heuristics are proposed and tested on a simple bi-objective 3-SAT problem. A first approach is based on a straightforward deformation of the survey-propagation equation to locally encode a Pareto trade-off. A simple heuristic is then tested, which combines an elimination procedure of clauses with the usual decimation of variables used in the survey propagation algorithm, and is able to sample different regions of the Pareto-front. In a second stage we study in more details the compliance of these deformed equations with basic belief-propagation (BP) properties. This lead us first to an explicit Markov random field of valid warning configuration, for which the survey-propagation equations are basic belief propagation equations. This observation is then generalized by defining a MRF for warnings configurations expected to approximate well the Pareto-front. The survey propagation equations associated to this new MRF are derived, allowing for consistent estimations of the Pareto-set on single problem instances. Numerical experiments on artificial problems up to 10^5 variables are presented and discussed.

Key-words: SAT, multi objective optimization, message passing, survey-propagation, Markov Random fields

* INRIA-Saclay, LRI Bât. 490, F-91405 Orsay(France)
† INRIA-Saclay, LRI Bât. 490, F-91405 Orsay(France)

Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université
4, rue Jacques Monod, 91893 ORSAY Cedex
Téléphone : +33 1 72 92 59 00

Problèmes 3-SAT Multi Critères par Passage de Messages

Résumé : Les algorithmes de passage de messages ont permis de résoudre des problèmes combinatoires difficiles avec pour résultats des progrès notables dans le domaine K-SAT aléatoire. Cependant, les résultats obtenus par survey-propagation sont limités à des problèmes SAT mono-critères alors que la plupart des problèmes réel sont multi objectifs. Une première approche au contexte multi objectif est proposée, dont le but est d'échantillonner le front de Pareto, c.a.d. l'ensemble des solutions non-dominées au sens de Pareto. Plusieurs heuristiques sont proposées et testées, sur un problème 3-SAT bi-objectifs. Une première approche est basée sur une déformation directe des équations survey-propagation afin d'encoder localement un compromis de Pareto. Une heuristique simple est ensuite testée qui combine une procédure d'élimination des clauses avec la décimation habituelle des variables utilisée dans l'algorithm survey-propagation, permettant d'échantilloner différentes régions du front de Pareto. Dans un deuxième temps, nous étudions en détails la compatibilité de ces équations avec les propiétés basique de belief-propagation. Ceci nous conduit d'abord à trouver un Champ Markovien aléatoire explicite sur les configurations de warnings pour lesquelles les équations de survey-propagation coincident avec les equations de belief-propagation. Cette observation est ensuite généralisée en définissant un champ Markovien aléatoire pour les configurations situées dans le voisinage du de l'ensemble de Pareto. Les équations de survey-propagation correspondantes que nous obtenons donne alors la possibilité d'estimer de fa**c**con cohérente du front de Pareto sur des exemple de problèmes. Des expériences numériques sur des problèmes artificiels atteignants 10^5 variables sont présentées et discutées.

Mots-clés : SAT, optimisation multi critères, passage de messages, survey propagation, Champs Markovien aléatoires

Contents

1 Introduction

Message passing algorithms based on the Belief-Propagation heuristic (BP) have been flourishing in various domains, and have proved very successful in particular in combinatorial optimization, to solve for example the random K-SAT problem with Survey-Propagation (SP) [1], or for clustering problems with affinity-propagation [2]. This allowed to address SAT problems involving a huge number of Boolean variables (up to one million). Furthermore, the link between BP and SP on the one hand, and the well-known DPLL procedure on the other hand [3], has been recently established [4], bridging the gap between the two communities of Statistical Physics and Operations Research.

However, it is well-known that most real-world problems are in fact multi objective and we are not aware of any work addressing multi objective problems with message-passing algorithms. Even more, most works in multi objective context deal with continuous variables, and most works in Multi Objective Combinatorial Problems (MOCO) address specific applications pertaining to scheduling or knapsack problems, rarely satisfiability problems. One noticeable exception is [5], where artificial multi objective problems are considered, but they involve at most 20 variables.

The goal of this paper is to extend the message-passing algorithms strategy from single- to multi objective context in the constraint satisfaction domain. The aim of multi objective optimization is to sample the Pareto set, i.e. the set of solutions that are not dominated in the Pareto sense in the decision space, and the Pareto front, i.e. the corresponding points in the objective space (each objective being a coordinate). The Pareto dominance relation defines a partial order on the decision space: a solution a dominates a solution b if a is better than b on at least one criteria, without being worse on any other. Therefore, on the Pareto set, a solution a cannot be worse than any other solution b on a given criteria, without being strictly better on at least one criteria. The knowledge of (a good approximation of) the Pareto set allows the user to make an informed decision, knowing exactly what will an increase on a given objective cost in terms of the other objectives.

For combinatorial optimization problems, message passing heuristics can be set up in principle, once a uniform measure is defined on the set of solutions, typically in the form of a Markov Random Field (MRF). Our guiding principle then for addressing the multi objective context is to search for a MRF, approximating well the Pareto set and at the same time suitable to run message passing algorithms. To illustrate this strategy, consider the following 2-objective problem: a random 3-SAT problem in the UNSAT phase, which set of clauses $\mathcal{F}$ is arbitrarily partitioned into two subsets $\mathcal{F}_0$ and $\mathcal{F}_1$, each one defining a sub 3-SAT problem in the SAT phase. The combination of both defines a 2-objective problem instance. Note that a possible alternative strategy referred as criteria aggregation, consists in optimizing a set of weighted combination of the two objectives using a single-objective optimizer.

However, this strategy can only sample the convex parts of the Pareto front, which does not include in general the whole front. This aggregated approach can nevertheless be useful for the sake of comparison. In particular, solutions to the MAX-SAT problem with equal weights pertains to the Pareto front.

The paper is organized as follows: in section 2 we define the benchmark problem and give a brief introduction to the survey-propagation algorithm and underlying assumptions. In section 3 we discuss how the Pareto dominance can be inserted locally into the survey-propagation equations, and how the Pareto front can be estimated on single problem instances. In section 5, a simple heuristic based on the modified equations is presented along with numerical results.

2 Multi Objective Random 3-SAT Benchmark Problem

2.1 Random 3-SAT problem

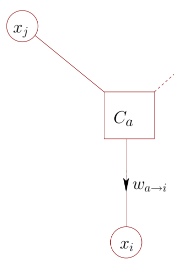
The 3-SAT problem is a decision problem involving a set $\mathcal{V}$ of N binary decision variables $x_i \in \{0,1\}, i = 1 \ldots N$ ($FALSE$ or $TRUE$), subjected to a conjunction of a set $\mathcal{F}$ of M constraints or clauses. Defined in conjunctive normal form the problem reads,

$$C_{\mathcal{V},\mathcal{F}} = \bigwedge_{a=1}^{M} C_a(x_a)$$

where $x_a = (x_i, x_j, x_k)$ is a subset of $\mathcal{V}$, with i, j and k in $\{1, \ldots, N\}$; clause C_a appears as the disjunction of three variables, like e.g.

$$C_a(x_a) = x_i \vee x_j \vee \bar{x}_k.$$

where each literal corresponds to a negated or non-negated variable. The clause is SAT if at least one of its literal is $TRUE$. The clause density $\alpha \stackrel{\text{def}}{=} M/N$ measures the difficulty of the problem. The random SAT is a family of problems indexed by this control parameter, a given instance being obtained by taking



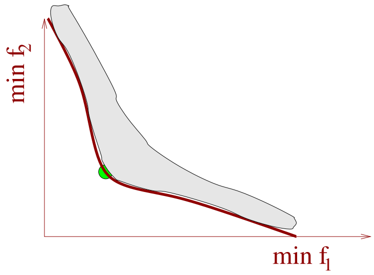


Fig. 2.1: Warning message send on the factor-graph (left). Approximate MRF associated to a bi-objective Pareto set (right), with the max-sat point represented.

at random the subset x_a of variables attached to any given clause a and the sign of each literal is also taken at random. The phase diagram of random K-SAT has been determined and refined over the years mainly with help of mean-field considerations[1, 6, 7, 8]. Various clustering phenomena taking place in the solutions space give its structure to the phase diagram. Schematically for $3-SAT$ we have in the thermodynamic limit:

- a sharp $SAT-UNSAT$ transition occurring at $\alpha = \alpha_c \simeq 4.267$, saying that the probability for the problem to be SAT drops discontinuously from 1 to 0;
- for $\alpha \leq 3.86$ there is in the statistical physics parlance[9], a replica symmetric (RS) SAT phase, corresponding to a giant cluster of nearby solutions;
- for $\alpha \geq 3.86$ either the 1-step (1-RSB) or the full step (f-RSB) of replica symmetric breaking phase occurs, corresponding to a single or many levels of clustering of the ground-state measure. The domain $\alpha \in [3.86, 4.267]$ is referred to as the hard SAT phase.
- for $\alpha \in [4.15, 4.39]$ the 1-RSB phase is stable, and the survey-propagation algorithm is based on this property.

2.2 Survey-Propagation Equation and Decimation Based Algorithm

Let us give here a brief overview of the survey-propagation equations and associated decimation algorithm (see [10] for details). On a single problem-instance, mean-field approach based on the cavity method[9] are translated into a set of equations: the survey-propagation equations whose fixed point solutions give statistical information on the variables, which in turn can be used to find solutions efficiently. The survey-propagation equations assume a 1-RSB phase in which solutions are grouped into well-separated clusters, these clusters being parametrized (presumably in a non-unique way) by a set of binary variables $w_{a\to i} \in \{0, 1\}$ called warning, attached to each link relating a clause a to a variable i on the factor graph[11] (see Figure 2.1). When a variable receives such a message it has to adopt the value requested by the clause sending this message. A given configuration of warnings is valid iff:

- no variable receives any contradictory warnings;
- a clause send a warning to one of its neighbours if its other neighbors received incompatible warnings with the requirement of that clause.

Fixing in a self-consistent way the values of these warnings is actually equivalent to run belief-propagation algorithm on a MRF associated to SAT assignments [6]. Let $J_{ai} \in \{-1, 1\}$ say whether a variable x_i is negated (-1) or not $(+1)$ in clause a and let $\tau_{aib} \stackrel{\text{def}}{=} \frac{1+J_{ai}J_{bi}}{2} \in \{0, 1\}$ indicate if clause a and b have compatible requirements ($\tau_{aib} = 1$) or not ($\tau_{aib} = 0$) w.r.t. variable i . The self-consistency rule for the warnings reads:

$$w_{a\to i} = \prod_{j\in a\setminus i} \Pi^{-}_{j\to a}(w_j) \tag{2.1}$$

with $w_j = \{w_{b\to j}, b \ni j\}$[1], and

$$\Pi^{0}_{i\to a}(w_i) \stackrel{\text{def}}{=} \prod_{b\ni i\setminus a} \bar{w}_{b\to i}, \tag{2.2}$$

$$\Pi^{+}_{i\to a}(w_i) \stackrel{\text{def}}{=} \prod_{b\ni i\setminus a} \big(\bar{w}_{b\to i} + \tau_{aib}\ w_{b\to i}\big) - \Pi^{0}_{i\to a}(w_i), \tag{2.3}$$

$$\Pi^{-}_{i\to a}(w_i) \stackrel{\text{def}}{=} \prod_{b\ni i\setminus a} \big(\bar{w}_{b\to i} + \bar{\tau}_{aib}\ w_{b\to i}\big) - \Pi^{0}_{i\to a}(w_i). \tag{2.4}$$

In the hard SAT phase, this schema is actually not working because of the clustering of solutions phenomena. The survey propagation algorithm find a uniform measure on the valid warning assignments by propagating instead the probability

$$\eta_{a\to i} = P(w_{a\to i} = 1),$$

called the survey. Assuming probabilistic independence of warnings sent to a given variable, the survey propagation equation then reads,

$$\eta_{a\to i} = \prod_{j\in a\setminus i} \frac{\Pi^{-}_{j\to a}(\eta_j)}{\Pi^{0}_{j\to a}(\eta_j) + \Pi^{+}_{j\to a}(\eta_j) + \Pi^{-}_{j\to a}(\eta_j)}. \tag{2.5}$$

where again η_j is the set of surveys received by j. The denominator here corresponds to a conditioning on non-contradictory warnings under the independent law defined by the set of surveys.

The fixed-point solution can then be used to simplify SAT formulas by fixing the most polarized variable. Iterating this procedure constitutes the SP-decimation algorithm, which ends when the fixed point degenerates with all surveys identically zero. At this point the reduced problem is expected to be very easy to solve with a local search algorithm.

[1] $j \in b$ is a shorthand notation expressing that j is neighbour to b

In the $UNSAT$ phase, the problem (aka $MAXSAT$) is instead to find configurations with lowest possible number of violated clauses. The equations (2.5) are not suitable in that case, although they converge up to a value of $\alpha \simeq 4.35$, yielding a fixed point with negative complexity (the log number of clusters of solutions). Introducing in the cavity equation a pseudo inverse-temperature y, the Legendre conjugate parameter to the derivative of the entropy w.r.t the energy[1], allows to get probabilistic information on states with positive energy. An efficient way of solving these mean-field equations on single-problem instances coupled with decimation and backtracking has been obtained, yielding the SP-Y algorithm[12].

3 SP Deformed Equations with Local Pareto Constraints

3.1 *Bi-objective 3-SAT benchmark and associated local Pareto criteria*

The bi-objective benchmark problem that we consider in this paper consists simply in having two sets of clauses $\mathcal{F}_0$ and $\mathcal{F}_1$ instead of a single one, while keeping a single set $\mathcal{V}$ of variables. For simplicity $\mathcal{F}_0$ and $\mathcal{F}_1$ are taken to be of equal size

$$M_1 = M_2 = M/2 \qquad \text{with} \qquad M/N < 2\alpha_c, \tag{3.1}$$

which means that each sub-problem $(\mathcal{V}, \mathcal{F}_\mu)$, $\mu \in \{1, 2\}$ taken independently is in the SAT phase while the junction of the two $(\mathcal{V}, \mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1)$ is in the $UNSAT$ phase.

To adapt the survey-propagation equations to this multi objective context we consider the Pareto dominance relation between solutions at the local level, by comparing two solutions separated by a single variable flip: we can say that a variable is Pareto optimal if under a flip it cannot increase the number of SAT clauses of one objective without strictly increasing the number of $UNSAT$ clauses for the other one.

With the chosen value of the clause density (3.1), each sub-problem taken alone can be made SAT, henceforth the Pareto set contains solutions for which one of the 2 sub-problem is SAT. This leads us to consider the ensemble of valid warning configuration in which a variable cannot receive contradictory warning emitted from the same sub-problem. We are looking for warning configurations which may have mutual conflicts between sub-problems, but for which internal conflicts, i.e., contradictory warnings send by clauses pertaining to the same sub-problem, are excluded. Then a variable may be in three different situations which all imply a local Pareto equilibrium:

- the variable is unconstrained, it does not receive any warning and can take either $TRUE$ or $FALSE$ value without modifying any of the objective.
- the variable receives at least one warning but without any contradiction, so that it takes the value obeying to the warnings.
- the variable receives at least one warning from $\mathcal{F}_0$ and $\mathcal{F}_1$ and these are contradictory. In that case the variable can chose to conform to either $\mathcal{F}_0$

or $\mathcal{F}_1$. Under a flip, one of the sub problem will lose at least one SAT clause while the other will gain at least one.

3.2 Deformed SP equations

To cope with this new specification, considering first the quantity $\Pi^c_{i\to a}$ associated to these type of contradictions:

$$\Pi^c_{i\to a}(w_i) = \Pi^{+-}_{i\to a}(w_i) + \Pi^{-+}_{i\to a}(w_i),$$

where

$$\begin{aligned}
\Pi^{+-}_{i\to a}(w_i) = &\Big[\prod_{\{b\ni i/a\}\cap\mathcal{F}_a}\big(\bar{w}_{b\to i} + \tau_{aib}w_{b\to i}\big)\big) - \prod_{\{b\ni i/a\}\cap\mathcal{F}_a}\bar{w}_{b\to i}\Big] \\
&\times\Big[\prod_{\{b\ni i/a\}\cap\bar{\mathcal{F}}_a}\big(\bar{w}_{b\to i} + \bar{\tau}_{aib}w_{b\to i}\big)\big) - \prod_{\{b\ni i/a\}\cap\bar{\mathcal{F}}_a}\bar{w}_{b\to i}\Big] \\
\Pi^{-+}_{i\to a}(w_i) = &\Big[\prod_{\{b\ni i/a\}\cap\mathcal{F}_a}\big(\bar{w}_{b\to i} + \bar{\tau}_{aib}w_{b\to i}\big)\big) - \prod_{\{b\ni i/a\}\cap\mathcal{F}_a}\bar{w}_{b\to i}\Big] \\
&\times\Big[\prod_{\{b\ni i/a\}\cap\bar{\mathcal{F}}_a}\big(\bar{w}_{b\to i} + \tau_{aib}w_{b\to i}\big)\big) - \prod_{\{b\ni i/a\}\cap\bar{\mathcal{F}}_a}\bar{w}_{b\to i}\Big]
\end{aligned}$$

represent warning configuration where i receive at least one a-compatible warning from sub-problem $\mathcal{F}_a$ containing a and one a-incompatible warning from the other sub-problem $\bar{\mathcal{F}}_a$, or vice versa. A variable, submitted to two incompatible requests from the two sub-problems has now the freedom to choose to which one it obeys. Averaging over this choice induces anyway some correlations between warnings which are difficult to handle, so we fix from the beginning the choice that will take each variable in case of a contradiction. Let $\theta_i \in \{0,1\}$ represent this binary choice. The rule to send a warning is then determined by the relation

$$w_{a\to i} = \prod_{j\in a\backslash a}\big(\Pi^-_{j\to a}(w_j) + \bar{\theta}_{ai}\Pi^c_{j\to a}(w_j)\big), \tag{3.2}$$

where $\theta_{ai} = \theta_a\theta_i + \bar{\theta}_a\bar{\theta}_i \ \in \{0,1\}$ if $\theta_a \in \{0,1\}$ gives the appartenance set $\mathcal{F}_a$ of a.

The survey propagation equations are then adapted as follows, by taking into account conflicting sets of warning but with some penalization factor q:

$$\eta_{a\to i} = \prod_{j\in a\backslash i}\frac{\Pi^-_{j\to a}(\eta_j) + q\bar{\theta}_{ai}\Pi^c_{j\to a}(\eta_j)}{\Pi^0_{j\to a}(\eta_j) + \Pi^+_{j\to a}(\eta_j) + \Pi^-_{j\to a}(\eta_j) + q\Pi^c_{j\to a}(\eta_j)}. \tag{3.3}$$

The difference with the basic survey-propagation scheme is that contradictions between warnings are allowed as long as they do not arise within a single problem component. For each variable experiencing a contradiction between warnings, the warnings configuration is weighted by a factor $q < 1$. The basic SP scheme is recovered in the limit where $q = 0$ while when $q = 1$ all warnings configurations with contradictions between sub-problems are taken into account with equal weight.

3.3 Complexity and Clause Elimination Criteria

Each value of q defines a statistical ensemble where contradictions are more or less filtered out. The best configuration are expected to be in the non-empty ensemble with lowest value $q*$. A simple criteria to determine $q*$ is based on the computation of the corresponding entropy of that ensemble,

$$q^* = \underset{q}{\text{argmin}}\{\Sigma[\eta(q)] = 0\},$$

where $\eta(q)$ denotes the set of surveys obtained for a given value of q and Σ denotes the complexity (i.e. the logarithm of the number of cluster solutions). Let $\tau_{ai} \stackrel{\text{def}}{=} \frac{1+J_{ai}}{2}$ and consider the quantities

$$\Pi_i^0(\eta_i) \stackrel{\text{def}}{=} \prod_{a\ni i} \bar{\eta}_{a\to i}, \qquad \Pi_i^+(\eta_i) \stackrel{\text{def}}{=} \prod_{a\ni i}\big(\bar{\eta}_{a\to i} + \tau_{ai}\eta_{a\to i}\big) - \Pi_i^0(\eta_i),$$

$$\Pi_i^-(\eta_i) \stackrel{\text{def}}{=} \prod_{a\ni i}\big(\bar{\eta}_{a\to i} + \bar{\tau}_{ai}\eta_{a\to i}\big) - \Pi_i^0(\eta_i),\ \ \Pi_i^c(\eta_i) \stackrel{\text{def}}{=} \Pi_i^{-+}(\eta_i) + \Pi_i^{+-}(\eta_i),$$

where $\bar{\eta}_{a\to i} \stackrel{\text{def}}{=} 1 - \eta_{a\to i}$ and $\Pi_i^{\pm\pm}$ are defined as products of $\Pi_i^{\pm}$ restricted to sub-problems 1 and 2. The complexity is expected to take the following form

$$\Sigma = \sum_a \log Z_a + \sum_i \big[(1-d_i)\log Z_i + E_i\big] \tag{3.4}$$

where

$$E_i = -\Pi_i^c \ \log q, \qquad Z_i = \Pi_i^0 + \Pi_i^- + \Pi_i^+ + q\Pi_i^c,$$

and Z_a expressing the probabilistic weight of each warning configuration w_a from the set of surveys η_a reads:

$$\begin{aligned}
Z_a \ &= \prod_{i\in a} Z_{i\to a} - \prod_{i\in a}\big(\Pi_{i\to a}^{-0} + \Pi_{i\to a}^{--} + q\Pi_{i\to a}^{-+}\big) \\
&+ (q^3-1)\Big[\prod_{i\in a}\Pi_{i\to a}^{-} - \prod_{i\in a}\big(\Pi_{i\to a}^{-0} + \Pi_{i\to a}^{--}\big)\Big] \\
&+ (q^3-q)\sum_{i\in a}\bar{\theta}_{ai}\Big[\Pi_{i\to a}^{c}\prod_{j\in a\backslash i}\Pi_{j\to a}^{-} - \Pi_{i\to a}^{-+}\prod_{j\in a\backslash i}\big(\Pi_{j\to a}^{-0} + \Pi_{j\to a}^{--}\big)\Big] \\
&+ (q^3-q^2)\sum_{i\in a}\Big[\Pi_{i\to a}^{-}\prod_{j\in a\backslash i}\bar{\theta}_{aj}\Pi_{j\to a}^{c} - \big(\Pi_{i\to a}^{-0} + \Pi_{i\to a}^{--}\big)\prod_{j\in a\backslash i}\bar{\theta}_{aj}\Pi_{j\to a}^{-+}\Big]
\end{aligned}$$

(omitting the argument η_i in the Π's) with

$$Z_{i\to a} \stackrel{\text{def}}{=} \Pi_{i\to a}^0 + \Pi_{i\to a}^- + \Pi_{i\to a}^+ + q\Pi_{i\to a}^c,$$

represent local partition functions attached to variables and factor node. The different subtractions terms to the bold product of independent statistical contributions, which appears in Z_a, correspond to the impossibility to have contradictions within the same sub-problem (first subtraction term) and the reweightings due to additional variables under contradiction. The other terms correspond to

cases where a is violated, so that all its neighbours variables are subjected to contradictions, while some were already without a and some where not. The q^3-1 term is for a causing to its neighbours to have contradiction, while there where no, without a. The q^3-q term is for a causing to two of its neighbours to become contradictory, while the 3rd one was already. And lastly the q^3-q^2 term corresponds to for a causing on additional neighbor to be contradictory while the two others were already under contradiction *because of* a.
Analogously we can express also the probability

$$P_a^v = q^3\frac{\Pi_a^c}{Z_a}\times\prod_{i\in a}\bar{\theta}_{ai}, \tag{3.5}$$

for a clause to be violated, with

$$\begin{aligned}\Pi_a^c \stackrel{\text{def}}{=} &\prod_{i\in a}\Pi_{i\to a}^c + \Bigl[\prod_{i\in a}\Pi_{i\to a}^- - \prod_{i\in a}(\Pi_{i\to a}^{-0}+\Pi_{i\to a}^{--})\Bigr]\\ &+\sum_{i\in a}\bar{\theta}_{ai}\Bigl[\Pi_{i\to a}^c\prod_{j\in a\backslash i}\Pi_{j\to a}^- - \Pi_{i\to a}^{-+}\prod_{j\in a\backslash i}\left(\Pi_{j\to a}^{-0}+\Pi_{j\to a}^{--}\right)\Bigr]\\ &+\sum_{i\in a}\Bigl[\Pi_{i\to a}^-\prod_{j\in a\backslash i}\bar{\theta}_{aj}\Pi_{j\to a}^c - \left(\Pi_{i\to a}^{-0}+\Pi_{i\to a}^{--}\right)\prod_{j\in a\backslash i}\bar{\theta}_{aj}\Pi_{j\to a}^{-+}\Bigr]\end{aligned}$$

We can distinguish between two contributions,

$$P_a^v = q^3\frac{\prod_{i\in a}\bar{\theta}_{ai}\Pi_{i\to a}^c}{Z_a} + \Delta P_a^v, \tag{3.6}$$

where the first contribution is coming from the environment of the clause, and the second term represents the direct impact of the clause, causing some new variables to be under contradiction. This quantity, ΔP_a^v will be useful when trying to identify which clauses are the most difficult to satisfy.

4 BP compliance

The equations presented so far, although having simple rules suffers from an important drawback which we describe now. Some compatibilities between surveys, at the basis of the BP schema are not satisfied, this preventing us from an exact evaluation of P_a^v as well as Σ, and henceforth a reliable estimation of the Pareto front. This motivates a closer investigation of the compliance of these equations with the basic belief propagation (BP) equations. This question has been addressed in various ways for SP, first in [13], using a dual formulation on an extended factor graph and in [14] by introducing the notion of cover. We propose here another connection holding directly at the level of warnings.

In general, BP yields 2 sets of exact or approximate marginals $\{b_i(x_i), i\in\mathcal{V}\}$ and $\{b_a(x_a), a\in\mathcal{F}\}$, called the beliefs, where x_i are the variables of a given problem and $x_a=\{x_i, i\in a\}$ the variables attached to a given factor. The convergence of BP enforces the compatibilities between beliefs:

$$b_i(x_i) = \sum_{\substack{x_j,\\ j\in a\backslash i}} b_a(x_a), \qquad \forall i\in a \tag{4.1}$$

for all factor $a \in \mathcal{F}$ if $\mathcal{F}$ is the set of factors of the underlying joint probability measure. In our context, at the level of the warning description, the variables are $w_i \stackrel{\text{def}}{=} \{w_{b\to i}; b \ni i\}$ and the factors correspond to the clauses, with corresponding attribute $w_a \stackrel{\text{def}}{=} \{w_{b\to i}; b \ni i, i \in a\}$, and the beliefs $b_i(w_i)$ and $b_a(w_a)$ may be obtained in principle from the surveys, but without any guaranty that the compatibility (4.1) holds. For instance, the computation of P_a^v may be performed in four different and in principle equivalent ways:

- by using the joint belief $b_a(w_a)$ yielding the form (3.6)
- by using the joint belief $b_i(w_i)$ and $\eta_{a\to i}$ for any of the three variables $i \in a$, and expressing the probability that a send a message to i while i has to conform to the sub-problem not containing a (which automatically implies that (and is in fact equivalent to) a is violated):

$$P_a^v = \eta_{a\to i}\bar{\theta}_{ai}\big(\Pi^{0-}_{i\to a}+\Pi^{+-}_{i\to a}\big)/\big(\Pi^0_{i\to a}+\Pi^-_{i\to a}+\Pi^+_{i\to a}+q\Pi^c_{i\to a}\big), \qquad \forall i \in a.$$

The problem is that the equivalence between these different estimations is not verified, because some correlations between warnings in w_a are not taken into account in $b_a(w_a)$ and among surveys. To cure this problem let us analyze this question in the SP context first.

4.1 The Case of SP

The mapping of SP to the standard BP schema has been addressed first in [13], using a dual formulation on an extended factor graph and then restated in [14] by introducing the notion of covers to model cluster of solutions on an extended variable space. Here we propose in fact to revisit this question, by establishing a link directly at the level of warnings. Consider that the attribute of a variable node i involved in the factor graph representation of this problem is the set of messages $w_i \stackrel{\text{def}}{=} \{w_{a\to i}, a \ni i\}$, while those of the factor nodes are the set of incoming warning on variables attached to a, namely $w_a \stackrel{\text{def}}{=} \{w_{b\to i}, i \in a, b \ni i\}$. The Markov random field associated to the uniform measure of valid warning configurations is then given by:

$$P(w) = \frac{1}{Z}\prod_{a\in\mathcal{F}} C_a(w_a)\prod_{i\in\mathcal{V}} C_i(w_i) \tag{4.2}$$

with

$$\begin{aligned}
C_i(w_i) &\stackrel{\text{def}}{=} \Pi_i^0(w_i) + \Pi_i^+(w_i) + \Pi_i^-(w_i),\\
C_a(w_a) &\stackrel{\text{def}}{=} \prod_{i\in a}\bar{w}_{a\to i}\big(\Pi^0_{i\to a}+\Pi^+_{i\to a}+\Pi^-_{i\to a}\big) - \prod_{i\in a}\bar{w}_{a\to i}\Pi^-_{i\to a}\\
&\qquad\qquad + \sum_{i\in a} w_{a\to i}\big(\Pi^0_{i\to a}+\Pi^+_{i\to a}\big)\prod_{j\in a\setminus i}\bar{w}_{a\to j}\Pi^-_{j\to a}.
\end{aligned}$$

$C_a(w_a)$ is defined in such a way to encode the rule (2.1) for emitting or not a message. Note that configurations in which a clause emits more that one message are excluded. Running the belief-propagation on this MRF results in the following message update rule:

$$m_{a\to i}(w_i) \propto \sum_{w_a/w_i} C_a(w_a) \prod_{j\in a/i} n_{j\to a}(w_j) \tag{4.3}$$

$$n_{i\to a}(w_i) = C_i(w_i) \prod_{b\ni i/a} m_{b\to i}(w_i) \tag{4.4}$$

This BP schema is well defined but potentially heavy because variables w_i are d_i-dimensional Boolean vectors, if d_i is the connectivity of w_i. A direct relation to survey-propagation is obtain from the following lemma.

Lemma 4.1. *Let $\eta_{a\to i} \in [0,1]$, the update rule (4.3) is stable with respect to the following parametrization of the message,*

$$\begin{aligned} m_{a\to i}(w_i) = {} & w_{a\to i}\big(\Pi^0_{i\to a}(w_i) + \Pi^+_{i\to a}(w_i)\big)\ \eta_{a\to i} \\ & + \bar{w}_{a\to i}\big(\Pi^0_{i\to a}(w_i) + \Pi^+_{i\to a}(w_i) + \Pi^-_{i\to a}(w_i)\big)(1-\eta_{a\to i}). \end{aligned} \tag{4.5}$$

with $\eta_{a\to i}$ satisfying the SP update rules.

Proof. Inserting the parametrization (4.5) in the left hand side of in (4.3) yields after a lengthy but straightforward computation:

$$\begin{aligned} m_{a\to i}(w_i) \propto {} & w_{a\to i}\big(\Pi^0_{i\to a}(w_i) + \Pi^+_{i\to a}(w_i)\big) \prod_{j\in a/i} \Pi^-_{j\to a}(\eta_j) \\ & + \bar{w}_{a\to i}\big(\Pi^0_{i\to a}(w_i) + \Pi^+_{i\to a}(w_i) + \Pi^-_{i\to a}(w_i)\big) \\ & \times \Big[\prod_{j\in a/i}\big(\Pi^0_{j\to a}(\eta_j) + \Pi^+_{j\to a}(\eta_j) + \Pi^-_{j\to a}(\eta_j)\big) - \prod_{j\in a/i} \Pi^-_{j\to a}(\eta_j)\Big]. \end{aligned}$$

Normalizing by $\prod_{j\in a/i}\big(\Pi^0_{j\to a}(\eta_j) + \Pi^+_{j\to a}(\eta_j) + \Pi^-_{j\to a}(\eta_j)\big)$ we end up with the SP update rules. ■

This insure in particular that the BP-based entropy formula is correct as well as the various probabilities associated to variables and factor nodes obtained from the surveys.

4.2 Generalization

The MRF associated to the uniform measure of valid warning configuration is given by:

$$P(w) = \frac{1}{Z} \prod_{a\in\mathcal{F}} C_a(w_a) \prod_{i\in\mathcal{V}} C_i(w_i) \tag{4.6}$$

with

$$C_i(w_i) \stackrel{\text{def}}{=} \Pi^0_i(w_i) + \Pi^+_i(w_i) + \Pi^-_i(w_i) + q\Pi^c_i(w_i)$$

and $C_a(w_a)$ enforcing the self-consistent rules (3.2) used to send or not a warning. The minimal parametrization of the messages to cope with this MRF involves now 3 real independent messages, instead of a single one for SP. These are the probability coefficients required to account for the 4 relevant states of a variable in this case, whether it receives a warning or not from a ($w_{a\to i}=1$ or $w_{a\to i}=0$) and whether it is forced to contradict a or not. (see table 1). We

	$w_{a\to i}=0$	$w_{a\to i}=1$
i SAT a	$x_{a\to i}$	$z_{a\to i}$
i UNSAT a	$y_{a\to i}$	$t_{a\to i}$

Tab. 1: Different states of variable i w.r.t clause a and associated surveys

denote by $x_{a\to i}$, $y_{a\to i}$, $z_{a\to i}$ and $t_{a\to i}$ the associated probabilistic surveys with

$$x_{a\to i}+y_{a\to i}+z_{a\to i}+t_{a\to i}=1$$

and let $\Pi^x_{i\to a}(w_i)$,$\Pi^y_{i\to a}(w_i)$, $\Pi^z_{i\to a}(w_i)$ and $\Pi^t_{i\to a}(w_i)$ the corresponding indicator functions on these states. These read:

$$\begin{aligned}
\Pi^x_{i\to a} &\stackrel{\text{def}}{=} \bar{w}_{a\to i}\Big(\Pi^0_{i\to a}+\Pi^+_{i\to a}+\theta_{ai}\Pi^{+-}_{i\to a}+\bar{\theta}_{ai}\Pi^{-+}_{i\to a}\Big)\\
\Pi^y_{i\to a} &\stackrel{\text{def}}{=} \bar{w}_{a\to i}\Big(\Pi^-_{i\to a}+\bar{\theta}_{ai}\Pi^{+-}_{i\to a}+\theta_{ai}\Pi^{-+}_{i\to a}\Big)\\
\Pi^z_{i\to a} &\stackrel{\text{def}}{=} w_{a\to i}\Big(\Pi^0_{i\to a}+\Pi^+_{i\to a}+\theta_{ai}\big(\Pi^{0-}_{i\to a}+\Pi^{+-}_{i\to a}\big)\Big)\\
\Pi^t_{i\to a} &\stackrel{\text{def}}{=} w_{a\to i}\bar{\theta}_{ai}\Big(\Pi^{0-}_{i\to a}+\Pi^{+-}_{i\to a}\Big).
\end{aligned}$$

Checking for compatibility by adding these projectors gives

$$\Pi^x_{i\to a}+\Pi^y_{i\to a}+\Pi^z_{i\to a}+\Pi^t_{i\to a}=\Pi^0_i+\Pi^+_i+\Pi^-_i+\Pi^c_i,$$

as it should. The clause constraints can be rewritten with help of these operators:

$$C_a(w_a)=\prod_{i\in a}\Pi^x_{i\to a}(w_i)+\sum_{i\in a}\Pi^y_{i\to a}(w_i)\prod_{j\in a/i}\Pi^x_{j\to a}(w_j) \tag{4.7}$$

$$+\prod_{i\in a}\Pi^t_{i\to a}(w_i)+\sum_{i\in a}\Pi^z_{i\to a}(w_i)\prod_{j\in a/i}\Pi^y_{j\to a}(w_j) \tag{4.8}$$

and the message is parametrized in the following way:

$$m_{a\to i}(w_i)=x_{a\to i}\Pi^x_{i\to a}(w_i)+y_{a\to i}\Pi^y_{i\to a}(w_i)+z_{a\to i}\Pi^z_{i\to a}(w_i)+t_{a\to i}\Pi^t_{i\to a}(w_i).$$

In turn the variable-to-clause messages $n_{i\to a}$ may be rewritten in the following manner:

$$\begin{aligned}
n_{i\to a}(w_i)=\Pi^0_{i\to a}(w_i)\big(\bar{w}_{a\to i}Q^0_{i\to a}+w_{a\to i}Q^1_{i\to a}\big)\ +\ \Pi^+_{i\to a}(w_i)Q^+_{i\to a}\\
+\bar{w}_{a\to i}\Pi^-_{i\to a}(w_i)Q^-_{i\to a}+qw_{a\to i}\Pi^{0-}_{i\to a}(w_i)Q^{0-}_{i\to a}\\
+q\bar{w}_{a\to i}\Pi^c_{i\to a}(w_i)Q^c_{i\to a}+qw_{a\to i}\Pi^{+-}_{i\to a}(w_i)Q^{+-}_{i\to a},
\end{aligned}$$

where Q are quantities depending on the incoming messages (see below). Inserting this again in the update rule (4.3) yields after a tedious but straightforward computation the following survey propagation equations:

$$x_{a\to i} \propto \prod_{j\in a/i}\big(Q^0_{j\to a}+Q^+_{j\to a}+q(\theta_{aj}Q^{+-}_{j\to a}+\bar\theta_{aj}Q^{-+}_{j\to a})\big)$$
$$+\sum_{\substack{j\in a/i\\ k\in a/(i,j)}}\big(Q^0_{j\to a}+Q^+_{j\to a}+q(\theta_{aj}Q^{+-}_{j\to a}+\bar\theta_{aj}Q^{-+}_{j\to a})\big)\big(Q^-_{k\to a}+q(\bar\theta_{ak}Q^{+-}_{k\to a}+\theta_{ak}Q^{-+}_{k\to a})\big)$$

$$y_{a\to i} \propto \prod_{j\in a/i}\big(Q^0_{j\to a}+Q^+_{j\to a}+q(\theta_{aj}Q^{+-}_{j\to a}+\bar\theta_{aj}Q^{-+}_{j\to a})\big)$$
$$+\sum_{\substack{j\in a/i\\ k\in a/(i,j)}}\big(Q^1_{j\to a}+Q^+_{j\to a}+q\theta_{aj}(Q^{0-}_{j\to a}+Q^{+-}_{j\to a})\big)\big(Q^-_{k\to a}+q(\bar\theta_{ak}Q^{+-}_{k\to a}+\theta_{ak}Q^{-+}_{k\to a})\big)$$

$$z_{a\to i} \propto \prod_{j\in a/i}\big(Q^-_{j\to a}+q(\bar\theta_{aj}Q^{+-}_{j\to a}+\theta_{aj}Q^{-+}_{j\to a})\big)$$

$$t_{a\to i} \propto \prod_{j\in a/i} q\ \bar\theta_{aj}\big(Q^{0-}_{j\to a}+Q^{+-}_{j\to a}\big)$$

with

$$Q^0_{i\to a}=\prod_{b\ni i/a} x_{b\to i}$$

$$Q^1_{i\to a}=\prod_{b\ni i/a}\big(\tau_{aib}\ x_{b\to i}+\bar\tau_{aib}\ y_{b\to i}\big)$$

$$Q^+_{i\to a}=\prod_{b\ni i/a}\big(\tau_{aib}\ x_{b\to i}+\bar\tau_{aib}\ y_{b\to i}+\tau_{aib}\ z_{b\to i}\big)-\prod_{b\ni i/a}\big(\tau_{aib}\ x_{b\to i}+\bar\tau_{aib}\ y_{b\to i}\big)$$

$$Q^-_{i\to a}=\prod_{b\ni i/a}\big(\bar\tau_{aib}\ x_{b\to i}+\tau_{aib}\ y_{b\to i}+\bar\tau_{aib}\ z_{b\to i}\big)-\prod_{b\ni i/a}\big(\bar\tau_{aib}\ x_{b\to i}+\tau_{aib}\ y_{b\to i}\big)$$

$$Q^{0-}_{i\to a}=\prod_{b\ni i/a}\big(x_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+y_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+\bar\theta_{ab}\bar\tau_{aib}(\bar\theta_{ai}z_{b\to i}+\theta_{ai}t_{b\to i})\big)$$
$$-\prod_{b\ni i/a}\big(x_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+y_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})\big)$$

$$\begin{aligned}
Q_{i\to a}^{+-} = &\Big[\prod_{b\ni i/a\cap\mathcal{F}_a}\big(x_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+y_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+\tau_{aib}(\theta_{ai}z_{b\to i}+\bar\theta_{ai}t_{b\to i})\big)\\
&-\prod_{b\ni i/a\cap\mathcal{F}_a}\big(x_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+y_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})\big)\Big]\\
&\times\Big[\prod_{b\ni i/a\cap\bar{\mathcal{F}}_a}\big(x_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+y_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+\tau_{aib}(\theta_{ai}z_{b\to i}+\bar\theta_{ai}t_{b\to i})\big)\\
&-\prod_{b\ni i/a\cap\bar{\mathcal{F}}_a}\big(x_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+y_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})\big)\Big]
\end{aligned}$$

and

$$\begin{aligned}
Q_{i\to a}^{-+} = &\Big[\prod_{b\ni i/a\cap\mathcal{F}_a}\big(x_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+y_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+\bar\tau_{aib}(\theta_{ai}z_{b\to i}+\bar\theta_{ai}t_{b\to i})\big)\\
&-\prod_{b\ni i/a\cap\mathcal{F}_a}\big(x_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+y_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})\big)\Big]\\
&\times\Big[\prod_{b\ni i/a\cap\bar{\mathcal{F}}_a}\big(x_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+y_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})+\tau_{aib}(\bar\theta_{ai}z_{b\to i}+\theta_{ai}t_{b\to i})\big)\\
&-\prod_{b\ni i/a\cap\bar{\mathcal{F}}_a}\big(x_{b\to i}(\tau_{aib}\bar\theta_{ai}+\bar\tau_{aib}\theta_{ai})+y_{b\to i}(\tau_{aib}\theta_{ai}+\bar\tau_{aib}\bar\theta_{ai})\big)\Big]
\end{aligned}$$

Once the a set of messages satisfying these equations is found, a certain number of quantities of algorithmic interest may be computed, like e.g. the probability P_i^c for a variable of being submitted to a contradiction or the probability P_a^v for a clause to be violated. These correspond to local marginals and can be computed exactly from the messages. Upon using the following quantities:

$$\begin{aligned}
Q_i^0 &\stackrel{\text{def}}{=} \prod_{a\ni i} x_{a\to i}\\
Q_i^+ &\stackrel{\text{def}}{=} \prod_{a\ni i}\big(x_{a\to i}\tau_{ai}+y_{a\to i}\bar\tau_{ai}+z_{a\to i}\tau_{ai}\big)-\prod_{a\ni i}\big(x_{a\to i}\tau_{ai}+y_{a\to i}\bar\tau_{ai}\big)\\
Q_i^- &\stackrel{\text{def}}{=} \prod_{a\ni i}\big(x_{a\to i}\bar\tau_{ai}+y_{a\to i}\tau_{ai}+z_{a\to i}\bar\tau_{ai}\big)-\prod_{a\ni i}\big(x_{a\to i}\bar\tau_{ai}+y_{a\to i}\tau_{ai}\big)\\
Q_i^c &\stackrel{\text{def}}{=} Q_i^{+-}+Q_i^{-+}
\end{aligned}$$

with

$$
\begin{aligned}
Q_i^{+-} &\stackrel{\text{def}}{=} \Big[\prod_{a\ni i\cap\mathcal{F}_i}\big(x_{a\to i}\tau_{ai}+y_{a\to i}\bar{\tau}_{ai}+z_{a\to i}\tau_{ai}\big)-\prod_{a\ni i\cap\mathcal{F}_i}\big(x_{a\to i}\tau_{ai}+y_{a\to i}\bar{\tau}_{ai}\big)\Big]\\
&\quad\times\Big[\prod_{a\ni i\cap\bar{\mathcal{F}}_i}\big(x_{a\to i}\bar{\tau}_{ai}+y_{a\to i}\tau_{ai}+t_{a\to i}\bar{\tau}_{ai}\big)-\prod_{a\ni i\cap\mathcal{F}_i}\big(x_{a\to i}\bar{\tau}_{ai}+y_{a\to i}\tau_{ai}\big)\Big]\\
Q_i^{+-} &\stackrel{\text{def}}{=} \Big[\prod_{a\ni i\cap\mathcal{F}_i}\big(x_{a\to i}\bar{\tau}_{ai}+y_{a\to i}\tau_{ai}+z_{a\to i}\bar{\tau}_{ai}\big)-\prod_{a\ni i\cap\mathcal{F}_i}\big(x_{a\to i}\bar{\tau}_{ai}+y_{a\to i}\tau_{ai}\big)\Big]\\
&\quad\times\Big[\prod_{a\ni i\cap\bar{\mathcal{F}}_i}\big(x_{a\to i}\tau_{ai}+y_{a\to i}\bar{\tau}_{ai}+t_{a\to i}\tau_{ai}\big)-\prod_{a\ni i\cap\bar{\mathcal{F}}_i}\big(x_{a\to i}\tau_{ai}+y_{a\to i}\bar{\tau}_{ai}\big)\Big]
\end{aligned}
$$

Concerning a given clause a, from the decomposition (4.7) of $C_a(w_a)$, the following quantities are as well useful:

$$
\begin{aligned}
Q_a^x &\stackrel{\text{def}}{=} \prod_{i\in a}\big(Q^0_{i\to a}+Q^+_{i\to a}+q(\theta_{ai}Q^{+-}_{i\to a}+\bar{\theta}_{ai}Q^{-+}_{i\to a})\big)\\
Q_a^{xy} &\stackrel{\text{def}}{=} \sum_{i\in a}\big(Q^-_{i\to a}+q(\bar{\theta}_{ai}Q^{+-}_{i\to a}+\theta_{ai}Q^{-+}_{i\to a})\big)\prod_{j\in a/i}\big(Q^0_{i\to a}+Q^+_{i\to a}+q(\theta_{ai}Q^{+-}_{i\to a}+\bar{\theta}_{ai}Q^{-+}_{i\to a})\big)\\
Q_a^{yz} &\stackrel{\text{def}}{=} \sum_{i\in a}\big(Q^1_{i\to a}+Q^+_{i\to a}+q\theta_{ai}(Q^{0-}_{i\to a}+Q^{+-}_{i\to a})\big)\prod_{j\in a/i}\big(Q^0_{i\to a}+Q^+_{i\to a}+q(\theta_{ai}Q^{+-}_{i\to a}+\bar{\theta}_{ai}Q^{-+}_{i\to a})\big)\\
Q_a^t &\stackrel{\text{def}}{=} \prod_{i\in a}q\ \bar{\theta}_{ai}\big(Q^{0-}_{i\to a}+Q^{+-}_{i\to a}\big)
\end{aligned}
$$

4.3 Complexity, energy and expected number of violated clauses

Once a fixed point of these equations is found, a certain number of probabilistic estimations can be made in a fully consitent way, owing to the underlying BP property. The complexity take again the form (3.4) with

$$
\begin{aligned}
Z_a &= Q_a^x+Q_a^{xy}+Q_a^{yz}+Q_a^t,\\
Z_i &= Q_i^0+Q_i^++Q_i^-+qQ_i^c,\\
E_i &= -\ Q_i^c\log q.
\end{aligned}
$$

For each variable there are two interesting quantities:

$$
\begin{aligned}
b_i^{fix} &= \frac{|Q_i^+-Q_i^-|}{Z_i},\\
P_i^c &= \frac{qQ_i^c}{Z_i},
\end{aligned}
$$

respectively the bias function quantifying how much a variable is polarized in one or in the other direction, and the probability of a variable to receive contradictory warnings.
For each clause, we can as well compute the probability

$$P_a^v = \frac{Q_a^t}{Z_a}.$$

that it is $UNSAT$. These expression can be used to estimate the Pareto front of a given problem instance by computing the expected number

$$\mathcal{E}_\mu(q,\theta) \stackrel{\text{def}}{=} \mathbb{E}\big[\#UNSAT_\mu\big] = \sum_{a\in\mathcal{F}_\mu} P_a^v, \qquad \mu \in \{0,1\}$$

of $UNSAT$ clause for each sub-problem, given the penalty q and a set $\theta = \{\theta_i, i \in \mathcal{V}\}$ of binary choices. Therefore, for a given choice of (q,θ) we can compute its corresponding estimate $(\mathcal{E}_0, \mathcal{E}_1, \Sigma)$. The set of non-dominated parameters choice regarding $(\mathcal{E}_0, \mathcal{E}_1)$ and for which $\Sigma \geq 0$ constitutes the estimation of the Pareto front corresponding to the 4-surveys equations.

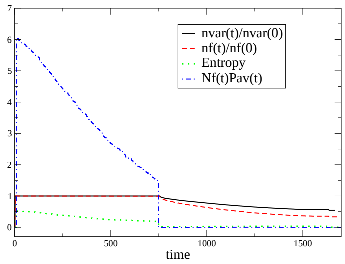
5 Numerical experiments

5.1 Sampling the Pareto set

We have run experiments using the deformed SP equation (3.3) to find Pareto solutions and compared them with max-sat solution obtained with SP-Y. Solution are generated by the following procedure:

- clause elimination: based on ΔP_a (3.6) with highest value, a small set of clauses are successively selected to be taken aside from the problem. $Nelim$, the total number of eliminated clauses is fixed in advance and in practice best results are obtained with a lower value than the one required to render the problem SAT.
- variable decimation: as in the original survey-propagation algorithm, variable with highest polarization are fixed sequentially, until the problem becomes paramagnetic or until convergence is lost.
- walksat is run onto the reduced problem a certain number of time to generate a cloud of solutions.

During both the elimination and the decimation stages, the penalty q is maintained at convergence threshold, which avoid to have an additional hyperparameter to tune, in addition to $Nelim$. This is done by trial and error, performing a gradual increase of q. The position of the solution found on the Pareto front, depends on how the clauses are selected in the elimination procedure. In our case it is implicitly determined by how the choice θ_i (see above in (3.3) of each variable i is set before letting survey-propagation converge. Among many possible heuristic, the one given best results so far, consists in eliminating n_0 clauses from problem $\mathcal{F}_0$ by letting $\theta_i = 1$ uniformly, and then to flip to $\theta_i = 0$ uniformly to eliminate $n_1 = Nelim - n_0$ clauses from objective $\mathcal{F}_1$.



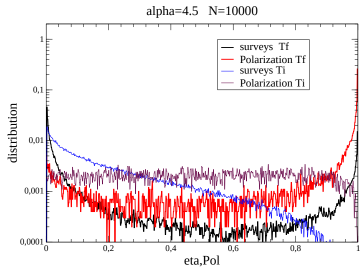


Fig. 5.2: Left panel: Fraction of active variables (nva), clauses (nfa), complexity and ΔP_a^v as function of time during the clause-elimination, variable-decimation process with fixed $\theta_i = 1$. Right panel: polarization of variable and surveys distribution at the beginning of the clause elimination (T_i) and at the end (T_f).

Let us make a few observations, concerning the clause elimination procedure: the distribution of surveys indicates that a pure state is obtained at the end of the process (see Figure.5.2), yielding variables which are either (almost) unpolarized or (almost) fully polarized. This is interesting both from the practical and theoretical view. It indicates that the landscape of the problem is progressively simplified, after each elimination of clause until a single valley remains. As a result if $Nelim$ is sufficiently large the simplified problem is very easy to solve. On Figure 5.3 one can see how the quality and position of solutions on the Pareto front depend on $Nelim$ and $Nelim_1$. Small value of $Nelim$ yield solution on the center while when $Nelim$ is increased the whole Pareto domain is scanned. The comparison with SP-Y is made by running it with backtracking and with various values of the pseudo inverse temperature y around the optimal $y*$ for which the complexity vanishes. The Pareto front which is obtained for the best tuning of $Nelim$ is not far from being optimal in the max-sat region when $\alpha < 4.4$ but the performance degrades when α increases, although it is stable with increased problem size N (see Figure. 5.3). Ideally, on this figure we should see the Pareto front entering the region below the Gardner energy [12], which is not the case yet. Clearly, although the convergence threshold is around $\alpha \simeq 6$, the relaxed Pareto criteria underlying our deformed survey-propagation equations (3.3) is problematic when going deeper inside the $UNSAT$ phase.

5.2 Pareto front estimation

With the 4-surveys equations of section 4.2, we can in principle estimate the Pareto-front of single problem instances in a consistent way, without providing any explicit solutions. The main difficulty comes from the large amount of possible choices for the θ_i's. Uniform setting $\theta_i = 0$ or $\theta_i = 1$ for all i yields estimates on the extreme points of the Pareto front, but in the bulk it is not clear how to fix these additional disorder variables. A random choice yields poor results, since we have at hand a potentially difficult optimization problem. The cheapest and best heuristic we have tested so far amounts to fix this variables dynamically, by switching uniformly θ between 0 and 1 for all variables but

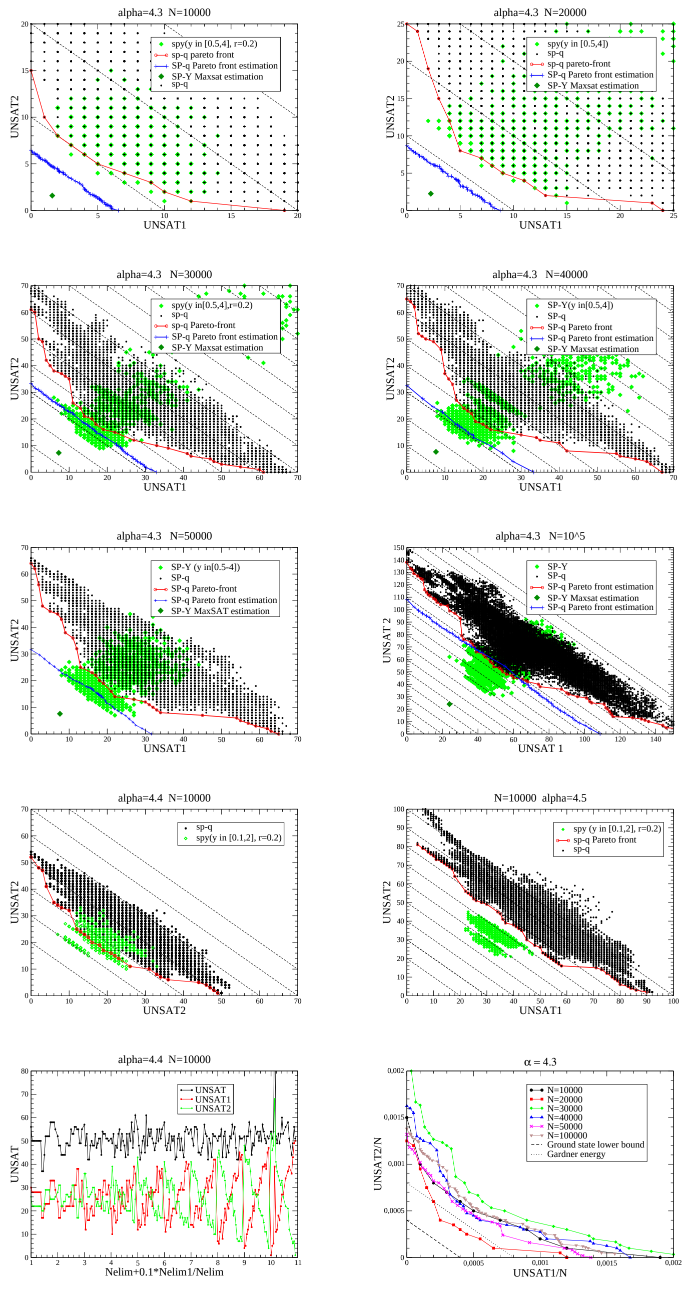


Fig. 5.3: Comparison between the Pareto front obtained with deformed SP equations, with solutions obtained from SP-Y (top panels). The Pareto front estimation and the SP-Y maxsat estimate are displayed for $\alpha = 4.3$. Results obtained for various combinations of $Nelim$ and $Nelim_0$ (bottom left) and rescaled Pareto-front obtained when N is varied at $\alpha = 4.3$ (bottom right).

the neighbors of the most problematic clauses i.e. having the highest value of P_a^v with some weighting between the two objective for ranking these clauses. In some sense this heuristic follows the same logic as of the clause elimination procedure discussed in section 5.1. A typical sample results is displayed on Figure.5.3 (left panel), in comparison with both the max-sat solutions found by SP-Y and the max-sat estimate from the SP-Y equations. In the center of the Pareto-front a factor of 2 between the 2 estimations is found, which is probably mainly due to the sub-optimal way used to fix the θ_i. What is observed also is that no solution with positive complexity can be found above $\alpha \simeq 4.4$, which is consistent with the phase diagram property of 3-SAT, indicating that the 1-RSB mean-field solution, underlying survey-propagation equation becomes unstable above $\alpha \simeq 4.39$.

6 Conclusion and perspectives

This paper has investigated the possibility to associate a MRF to the Pareto set of a specific multi objective problem example. A way to estimate this set has been proposed, as well as a simple heuristic able to sample it with reasonably good performance. Still, a gap with an optimal performance remains which may have several possible origins:

a simplifying assumption in the MRF definition, and an additional simplification in the SP equation used in Section 5.1; no backtracking techniques used in the elimination and decimation stage of the procedure. These issues requires additional work to be tackled, but more generally from this study we can foresee how our approach could be generalized to more than two objectives and other type of multi objective combinatorial problems. Any consistent message passing equations (in the sense discussed in section 4.2) like the one used in SP-Y or the one proposed in section 4.2 contains useful information on the clauses, which can be used in principle in an elimination procedure. This procedure, is seen to be quite efficient in our case and could be probably improved with backtracking techniques. Additionally it is completely generic for constraint satisfaction problems, so the idea would be to use it in combination with single objective message passing optimizers, as a basic tool to sample Pareto front of multi objective constraint satisfactions problems with many different objectives.

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Centre de recherche INRIA Saclay – Île-de-France
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